

The Z-cubes: a hypercube variant with small diameter

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Abstract

This paper introduces a new variant of hypercubes, which we call Z-cubes. The n -dimensional Z-cube H_n is obtained from two copies of the $(n - 1)$ -dimensional Z-cube H_{n-1} by adding a special perfect matching between the vertices of these two copies of H_{n-1} . We prove that the n -dimensional Z-cubes H_n has diameter $(1 + o(1))n/\log_2 n$. This greatly improves on the previous known variants of hypercube of dimension n , whose diameters are all larger than $n/3$. Moreover, any hypercube variant of dimension n is an n -regular graph on 2^n vertices, and hence has diameter greater than $n/\log_2 n$. So the Z-cubes are optimal with respect to diameters, up to an error of order $o(n/\log_2 n)$. Another type of Z-cubes $Z_{n,k}$ which have similar structure and properties as H_n are also discussed in the last section.

1 Introduction

Multiprocessor interconnection networks can be represented by graphs, where vertices represent processors and edges represent links between processors. For the processors to communicate efficiently, it is desired that the networks have small communication delay, i.e., the graphs have small diameter.

The hypercube network is one of the most popular interconnection networks and has been used in both the Intel iPSC and the NCUBE/10 systems. The n -dimensional hypercube Q_n is a graph whose vertices are binary strings of length n , with x, y adjacent if and only if x and y differ in exactly one bit. Alternately, we have $Q_1 = K_2$ with vertices 0 and 1, and for $n \geq 2$, Q_n is obtained from two copies of Q_{n-1} , $0Q_{n-1}$ and $1Q_{n-1}$, by adding an edge connecting $0x$ and $1x$ for every $x \in Q_{n-1}$. The popularity of hypercubes is due to its simple structure, relatively small diameter and small vertex degree.

However, hypercubes do not have the smallest diameter for its resource. Many variants of hypercubes have been introduced and studied in the literature, including twisted cubes [1], Mobius cubes [2], cross cubes [4], etc. The various n -dimensional cubes have binary strings of length n as vertices, they are n -regular, and have hierarchical structure, i.e., constructed from lower dimensional cubes by adding appropriate edges [3]. A main feature of the above mentioned hypercube variants is that they have smaller diameter: These n -dimensional cubes all have diameter about half of that of the n -dimensional hypercube.

A natural question is whether there are n -dimensional cubes with smaller diameter.

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In [8], a new hypercube variant, the spined cube, was introduced. It was proved in [8] that the n -dimensional spined cube has diameter about $n/3$. Up to now, this is the only hypercube variant which has diameter smaller than $n/2$. Is it possible to reduce further the diameter?

In this paper, by generalizing the construction in [8], we introduce a new variant of hypercubes H_n , which we call the Z-cubes. In [6], a graph G is called an n -dimensional *bijective connection graph* (abbreviated as BC graph) if $V(G)$ can be partitioned into $V_0 \cup V_1$, each $G[V_i]$ is an $(n-1)$ -dimensional BC graph, and the edges between V_0 and V_1 is a perfect matching of G (with $G = K_2$ when $n = 1$). Many variants of hypercubes (including hypercubes itself) are special families of BC graphs. Our Z-cubes is also a special family of BC graphs. For $n = 1$, $H_1 = K_2$. For $n \geq 2$, H_n is obtained from the disjoint union of two copies of H_{n-1} by adding a special perfect matching between vertices of these two copies of H_{n-1} . We shall prove that H_n has diameter $(1 + o(1))\frac{n}{\log_2 n}$.

As any n -dimensional hypercube variant is n -regular and has 2^n vertices, easy counting shows that they have diameter larger than $n/\log_2 n$. So the n -dimensional Z-cube is near-optimal in the sense of diameter.

This paper is organized as follows: In Section 2, we present the construction of the Z-cube H_n and prove that for $n \geq 3$, H_n is Hamiltonian connected. In Section 3, we prove an upper bound for the diameter of H_n . In Section 4, we raise some open questions, and also present another kind of Z-cubes, $Z_{n,k}$. For each fixed integer k , $Z_{n,k}$ is a family of Z-cubes, whose diameter is bounded from above by $n/(k+1) + 2^k$. By taking $k = \lceil \log_2 n - 2 \log_2 \log_2 n \rceil$, the resulting Z-cube $Z_{n,k}$ has diameter $(1 + o(1))n/\log_2 n$. These Z-cubes have simpler structure than H_n . The disadvantage is that to achieve the bound $(1 + o(1))n/\log_2 n$, for different n , we need to start with different lower dimensional cubes.

2 The Z-cube H_n

Denote by Z_2^n the set of binary strings of length n . For $x, y \in Z_2^n$, $x \oplus y$ denotes the sum of x and y in the group Z_2^n , i.e., $(x \oplus y)_i = x_i + y_i \pmod{2}$ (for $x \in Z_2^n$, denote by x_i is the i th bit of x).

If x is a binary string of length n_1 and y is a binary string of length n_2 , then xy is the *concatenation* of x and y , which is a binary string of length $n_1 + n_2$. If Z is a set of binary strings, then let $xZ = \{xy : y \in Z\}$.

For $x \in Z_2^n$, and for $1 \leq i < j \leq n$, denote by $x[i, j]$ the binary string $x_i x_{i+1} \dots x_j$.

Let κ be the integer function defined as follows:

$$\kappa(n) = \begin{cases} 0, & \text{if } n=1, \\ \max\{1, \lceil \log_2 n - 2 \log_2 \log_2 n \rceil\}, & \text{otherwise.} \end{cases}$$

We define a permutation ϕ of binary strings as follows:

Definition 2.1. Assume $x \in Z_2^n$. Then $\phi(x) \in Z_2^n$ is the binary string such that

$$\begin{aligned} \phi(x)[1, \kappa(n)] &= x[1, \kappa(n)] \oplus x[n - \kappa(n) + 1, n], \\ \phi(x)[\kappa(n) + 1, n] &= x[\kappa(n) + 1, n]. \end{aligned}$$

Note that the restriction of ϕ to Z_2^n is indeed a permutation of Z_2^n , with $\phi^2(x) = x$.

Now we are ready to define Z-cube H_n .

Definition 2.2. If $n = 1$ then $H_n = K_2$, with vertices 0 and 1. Assume $n \geq 2$. Then H_n is obtained from two copies of H_{n-1} , $0H_{n-1}$ and $1H_{n-1}$, by adding edges connecting $0x$ and $1\phi(x)$ for all $x \in H_{n-1}$.

The vertex set of H_n is Z_2^n . For convenience, we use H_n to denote its vertex set as well. Thus for $\theta \in \{0, 1\}$, $\theta H_n = \theta Z_2^n = \{\theta x : x \in Z_2^n\} \subseteq Z_2^{n+1}$.

The following observation follows easily from the definition.

Observation 2.3. For $1 \leq q < n$, for $a \in Z_2^q$, the subset aZ_2^{n-q} induces a copy of H_{n-q} .

A walk W in H_n is viewed as a sequence of vertices of H_n . If $W = (v_1, v_2, \dots, v_m)$, then for $a \in Z_2^t$, $aW = (av_1, av_2, \dots, av_m)$ is a walk in H_{n+t} (here av_j is the concatenation of two binary strings). For two walks W_1, W_2 in H_n , if the last vertex of W_1 is adjacent to the first vertex of W_2 , then the concatenation of W_1 and W_2 , denoted by $W_1 \cup W_2$ is also a walk in H_n .

It follows from the definition that $H_1 = K_2, H_2 = C_4$ and H_3 is as depicted in Figure 1 below.

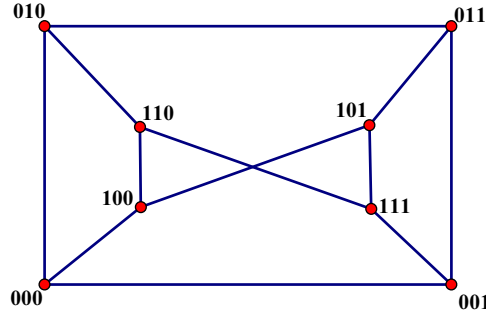


Figure 1: The graph H_3 .

A graph is called *Hamiltonian connected* if any two vertices are connected by a Hamiltonian path.

Lemma 1. For $n \geq 3$, the Z -cube H_n is Hamiltonian connected.

Proof. It is easy to check that H_3 is Hamiltonian connected. From this, it is easy to derive that for any $n \geq 3$, H_n is Hamiltonian connected. Indeed, assume $n \geq 4$ and H_{n-1} is Hamiltonian connected. For $x, y \in H_n$, if $x = 0x', y = 0y' \in 0H_{n-1}$ then let P be a Hamiltonian path of H_{n-1} connecting x' and y' . Let z' be the neighbor of x' in P , and let $P'' = P[z', y'] = P - x$. Let P' be a Hamiltonian path in H_{n-1} connecting $\phi(x')$ and $\phi(z')$. Then the path start with the edge $(0x', 1\phi(x'))$, followed by $1P'$, followed by the edge $(1\phi(z'), 0z')$ and then followed by $0P''$ is a Hamiltonian path of H_n connecting x and y . If $x = 0x' \in 0H_{n-1}$ and $y = 1y' \in 1H_{n-1}$, then one can find a Hamiltonian path connecting x and y by starting from x , pass through all vertices of $0H_{n-1}$, then pass through all vertices of $1H_{n-1}$ and ending at y . By induction hypothesis, the paths needed in $0H_{n-1}$ and $1H_{n-1}$ exist. \square

3 The diameter of H_n

Observe that for $x \in H_n$, a neighbor y of x in H_n is obtained from x by changing a number of bits lying in an interval of the form $[i, i + \kappa(n - i)]$ for some $i \leq n$. As $\kappa(n - i) \leq \kappa(n)$, for $x \in H_n$, the distance between x and its complement \bar{x} is at least $\lceil n/(\kappa(n) + 1) \rceil$.

In the following, we give an upper bound on the diameter for H_n . It follows from the definition that for any $j \geq 2$, $\kappa(j)$ is equal to either $\kappa(j - 1)$ or $\kappa(j - 1) + 1$. For a positive integer n , let

$$S(n) = \{j \leq n : \kappa(j) = \kappa(j - 1) + 1\}$$

and let

$$\sigma_n = \sum_{j \in S(n)} \frac{j}{(\kappa(j))^2}.$$

Theorem 3.1. *The graph H_n defined above has diameter at most $\frac{n}{\kappa(n)+1} + \sigma_n + 2^{\kappa(n)} + \kappa(n)$.*

Proof. For $n \leq 3$, this is obvious. Assume $n \geq 4$. For a positive integer $k \leq n$, a k -robust walk in H_n is a walk W such that for each $z \in Z_2^k$, there is a vertex $v \in W$ with $v[n - k + 1, n] = z$. Observe that if $k' < k$, then a k -robust walk in H_n is also a k' -robust walk. To prove Theorem 3.1, it suffices to prove the following claim.

Claim 3.2. *For any $n \geq k \geq \kappa(n)$, for any $x, y \in H_n$ (not necessarily distinct), there is a k -robust walk from x to y of length at most $n/(\kappa(n) + 1) + \sigma_n + 2^k + k$.*

Proof. We prove this claim by induction on n . For $k \leq n \leq 2k$, we claim that there is a k -robust walk of length $n - k + 2^k$ connecting x and y . If $n = k$, then let W be a Hamiltonian path in H_n connecting x and y , which has length at most 2^k . Assume $n \geq k + 1$ and the claim is true for $n - 1$. If x, y are in the same copy of H_{n-1} , then the conclusion follows from the induction hypothesis. Assume $x = 0x' \in 0H_{n-1}$ and $y = 1y' \in 1H_{n-1}$. By induction hypothesis, there is a k -robust walk W of length $n - 1 - k + 2^k$ in H_{n-1} connecting $\phi(x')$ and y' . Then the walk start from the edge $(0x', 1\phi(x'))$ followed by $1W$ is a k -robust walk of length at most $n - k + 2^k$ in H_n connecting x and y .

Assume $n \geq 2k + 1$ and the claim above holds for $n' < n$. By definition, H_n is obtained from two copies of H_{n-1} , $0H_{n-1}$ and $1H_{n-1}$, by adding an edge $0x \sim 1\phi(x)$ for each $x \in H_{n-1}$. If x, y belong to the same copy of H_{n-1} , then we are done by induction hypothesis. Assume $x \in 0Q_{n-1}$ and $y \in 1Q_{n-1}$.

Let $a = x[2, \kappa(n) + 1]$, $x' = x[\kappa(n) + 2, n]$, $b = y[2, \kappa(n) + 1]$ and $y' = y[\kappa(n) + 2, n]$. So $x = 0ax', y = 1by'$ and $x', y' \in H_{n-\kappa(n)-1}$.

By induction hypothesis, there is a k -robust walk W in $H_{n-\kappa(n)-1}$ from x' to y' of length at most

$$\frac{n - \kappa(n) - 1}{\kappa(n - \kappa(n) - 1) + 1} + \sigma_{n-\kappa(n)-1} + 2^k + k.$$

Let $c = a \oplus b$. Since W is a k -robust walk, there is a vertex $v \in W$ with $v[n - \kappa(n) + 1, n] = c$. Let $W_1 = W[x', v]$ be the subwalk of W from x' to v , and $W_2 = W[v, y']$ be the subwalk of W from v to y' . Then $0aW_1$ is a walk in H_n from $x = 0ax'$ to $0av$, and $1bW_2$ is a walk in H_n from $1bv$ and $1by' = y$. Since

$$\phi(av)[1, \kappa(n)] = a \oplus v[n - \kappa(n) + 1, n] = a \oplus c = b$$

and

$$\phi(av)[\kappa + 1, n] = v,$$

we have $\phi(av) = bv$ and hence $0av$ is adjacent to $1bv$ in H_n . Now the concatenation $0aW_1 \cup 1bW_2$ of $0aW_1$ and $1bW_2$ is a k -robust walk in H_n connecting x and y , whose length is

$$|W| + 1 \leq \frac{n - \kappa(n) - 1}{\kappa(n - \kappa(n) - 1) + 1} + \sigma_{n - \kappa(n) - 1} + 2^k + k + 1.$$

It remains to show that

$$\frac{n - \kappa(n) - 1}{\kappa(n - \kappa(n) - 1) + 1} + \sigma_{n - \kappa(n) - 1} + 2^k + k + 1 \leq \frac{n}{\kappa(n) + 1} + \sigma_n + 2^k + k.$$

I.e.,

$$\frac{n - \kappa(n) - 1}{\kappa(n - \kappa(n) - 1) + 1} + \sigma_{n - \kappa(n) - 1} + 1 \leq \frac{n}{\kappa(n) + 1} + \sigma_n.$$

It follows easily from the definition that $\kappa(n - \kappa(n) - 1)$ is either equal to $\kappa(n)$ or equal to $\kappa(n) - 1$. In the former case, we have $\sigma_n = \sigma_{n - \kappa(n) - 1}$, and hence

$$\frac{n - \kappa(n) - 1}{\kappa(n - \kappa(n) - 1) + 1} + \sigma_{n - \kappa(n) - 1} + 1 = \frac{n}{\kappa(n) + 1} + \sigma_n.$$

In the later case,

$$\sigma_n = \sigma_{n - \kappa(n) - 1} + \frac{j}{(\kappa(j))^2}$$

for some $n - \kappa(n) \leq j \leq n$, and hence

$$\sigma_n \geq \sigma_{n - \kappa(n) - 1} + \frac{n - \kappa(n)}{\kappa(n)^2}.$$

Therefore,

$$\begin{aligned} \frac{n - \kappa(n) - 1}{\kappa(n - \kappa(n) - 1) + 1} + \sigma_{n - \kappa(n) - 1} + 1 &\leq \frac{n - \kappa(n) - 1}{\kappa(n)} + \sigma_n - \frac{n - \kappa(n)}{\kappa(n)^2} + 1 \\ &= \frac{n - \kappa(n) - 1}{\kappa(n) + 1} + \frac{n - \kappa(n) - 1}{\kappa(n)(\kappa(n) + 1)} + \sigma_n - \frac{n - \kappa(n)}{\kappa(n)^2} + 1 \\ &\leq \frac{n}{\kappa(n) + 1} + \sigma_n. \end{aligned}$$

□

This completes the proof of Theorem 3.1. □

Corollary 3.3. *For any positive integer n , the Z -cube H_n has diameter at most $(1 + o(1))\frac{n}{\log_2 n}$.*

Proof. By Theorem 3.1, H_n has diameter at most

$$\begin{aligned}
& \frac{n}{\kappa(n) + 1} + \sigma_n + 2^{\kappa(n)} + \kappa(n) \\
& \leq \frac{n}{\log_2 n} \frac{\log_2 n}{\log_2 n - 2 \log_2 \log_2 n + 1} + \sigma_n + 2^{\kappa(n)} + \kappa(n) \\
& = (1 + o(1)) \frac{n}{\log_2 n} + \sigma_n + 2^{\kappa(n)} + \kappa(n).
\end{aligned}$$

As $\kappa(n) \leq \log_2 n - 2 \log_2 \log_2 n + 1$, we have

$$\begin{aligned}
2^{\kappa(n)} + \kappa(n) & \leq 2^{\kappa(n)+1} \\
& \leq \frac{4n}{(\log_2 n)^2} \\
& = o(n/\log_2 n).
\end{aligned}$$

To prove that H_n has diameter at most $(1+o(1))n/\log_2 n$, it remains to show that $\sigma_n = o(n/\log_2 n)$. By definition,

$$\sigma_n = \sum_{j \in S(n)} \frac{j}{(\kappa(j))^2} = \sum_{i=1}^{\kappa(n)} \frac{\kappa^{-1}(i)}{i^2}, \quad (1)$$

here we let $\kappa^{-1}(i) = \min\{j : \kappa(j) = i\}$. As $\log_2 i - 2 \log_2 \log_2 i \leq \kappa(i)$, we have

$$i \leq (\log_2 i)^2 2^{\kappa(i)}. \quad (2)$$

There is a constant c such that for $i \geq c$, $\frac{1}{2} \log_2 i \geq 2 \log_2 \log_2 i$. Plug this into the definition of κ , we have

$$\log_2 i \leq 2\kappa(i). \quad (3)$$

Plug (3) into (2), we have

$$i \leq (2\kappa(i))^2 2^{\kappa(i)}. \quad (4)$$

So for $i \geq c$,

$$\kappa^{-1}(i) \leq 4i^2 2^i. \quad (5)$$

Plug (5) into (1), we have

$$\sigma_n \leq 4 \sum_{i=c+1}^{\kappa(n)} 2^i + C,$$

where $C = \sum_{i=1}^c \frac{\kappa^{-1}(i)}{i^2}$ is a constant. Hence

$$\sigma_n \leq 2^{\kappa(n)+3} + C \leq 16 \frac{n}{(\log_2 n)^2} + C = o(n/\log_2 n).$$

This completes the proof of the corollary. \square

4 Some questions and discussions

The upper bound on the diameter of H_n given in Theorem 3.1 is probably not tight. One natural problem is to determine the diameter of H_n .

Question 4.1. *What is the diameter of H_n ?*

One nice property of the hypercube is that one can easily find a shortest path between any two vertices. However, it seems not easy to find the shortest path between two vertices in H_n .

Question 4.2. *Is there an algorithm that finds a shortest path between any two vertices of H_n in time polynomial of n ?*

Since the diameter of H_n is much smaller than that of the hypercube Q_n , it is perhaps good enough to have a quick algorithm that finds a short path (not necessarily the shortest) path between two vertices. For this purpose, it is also desirable to have knowledge on the average distance between two vertices of H_n .

Question 4.3. *What is the average distance between two vertices of Q_n ?*

We have shown that the Z-cube H_n is Hamiltonian connected for $n \geq 3$. Fault tolerance has been studied a lot for various variants of hypercubes [5]. The same question is also interesting for Z-cubes.

Question 4.4. *What is the minimum number of vertices and/or edges whose deletion results in a non-Hamiltonian graph?*

Not many automorphisms of H_n are known. It is unknown if H_n is vertex-transitive.

Question 4.5. *What is the automorphism group of H_n ?*

One can define a variant of hypercube which is similar to H_n , but have simpler structure. The precise definition is as follows:

For a fixed non-negative integer k , we define a permutation ϕ_k of binary strings as follows:

Definition 4.6. *Assume $x \in Z_2^n$. Then $\phi_k(x) \in Z_2^n$ is the binary string such that*

- *If $n \leq k$, then $\phi_k(x) = x$.*
- *If $k + 1 \leq n \leq 2k$, then*

$$\begin{aligned}\phi_k(x)[1, n - k] &= x[1, n - k] \oplus x[k + 1, n], \\ \phi_k(x)[n - k + 1, n] &= x[n - k + 1, n].\end{aligned}$$

- *If $n \geq 2k + 1$, then*

$$\begin{aligned}\phi_k(x)[1, k] &= x[1, k] \oplus x[n - k + 1, n], \\ \phi_k(x)[k + 1, n] &= x[k + 1, n].\end{aligned}$$

We define a family of Z-cubes $Z_{n,k}$ as follows:

Definition 4.7. If $n = 1$ then $Z_{n,k} = K_2$, with vertices 0 and 1. Assume $n \geq 2$. Then $Z_{n,k}$ is obtained from two copies of $Z_{n-1,k}$, $0Z_{n-1,k}$ and $1Z_{n-1,k}$, by adding edges connecting $0x$ and $1\phi_k(x)$ for all $x \in Z_{n-1,k}$.

A similar (but simpler) argument as the proof of Theorem 3.1 shows that the graph $Z_{n,k}$ has diameter at most $n/(k+1) + 2^k$.

For each n , let $Z_n^* = Z_{n,\kappa(n)}$. Then it is easy to verify (with an argument simpler than the proof of Corollary 3.3) that Z_n^* has diameter at most

$$\left(1 + \frac{2 \log_2 \log_2 n + 1}{\log_2 n - 2 \log_2 \log_2 n - 1} + \frac{1}{\log_2 n}\right) \frac{n}{\log_2 n} = (1 + o(1)) \frac{n}{\log_2 n}.$$

Although both Z-cubes H_n and Z_n^* have diameter $(1 + o(1))n/\log_2 n$, the upper bound for the diameter of Z_n^* obtained above is smaller than the upper bound for the the diameter of H_n in Theorem 3.1. So we may prefer the Z-cube Z_n^* to H_n , if we aim to construct a cube of a fixed dimension.

The disadvantage of Z_n^* is that for each n , to construct a Z-cube Z_n^* , we start with a special k from the very beginning. If we want to increase the dimension of the cube later, we may need to start the construction from a larger k . So we cannot just expand the existing cube. Instead we need to start the construction process from the very beginning. Nevertheless, if we fix an integer k at the beginning, as n goes to infinity, the diameter of $Z_{n,k}$ is at most $n/(k+1) + 2^k$, which is probably good enough for many purposes. At least it is better than the earlier variants of hypercubes, for the sake of diameter.

Also, it seems that the Z-cubes $Z_{n,k}$ have simpler structure than H_n , and it is probably easier to design a routing algorithm in $Z_{n,k}$. Nevertheless, the questions asked above are also open for the Z-cubes $Z_{n,k}$: (1) What is the diameter of $Z_{n,k}$? (2) Is there an algorithm that finds a shortest path between any two vertices of $Z_{n,k}$ in time polynomial in n ? (3) What is the average distance between two vertices of $Z_{n,k}$? (4) What is the minimum number of vertices and/or edges whose deletion results in a non-Hamiltonian graph? (5) What is the automorphism group of H_n ? The Z-cubes are new variants of hypercubes. Many problems about hypercubes and their variants should be interesting for Z-cubes as well.

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